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Estimates for the Szegö Kernel on Decoupled Domains*

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0. INTRODUCTION

In this paper we prove some L^p, $\mathcal{H}^p$, and Hölder estimates for the Szegö projection operator on decoupled domains in $\mathbb{C}^n$. First, let us recall what it means that a domain is decoupled and of finite type near a point of its boundary (the geometry of such a domain has been studied in [Mc1]).

DEFINITION 0.1. Let $\Omega \subset\subset \mathbb{C}^n$ be a regular pseudoconvex domain. Then, Ω is said to be decoupled of finite type near $\zeta \in \partial\Omega$ if there exists a holomorphic coordinate system $(z_1, \ldots, z_n)$ mapping ζ onto 0 and a neighborhood U_ζ of ζ onto a neighborhood U of 0 and smooth, subharmonic, but not harmonic, functions $\{f_k\}_{\{k=1,\ldots,n-1\}}$, $f_k : \mathbb{C} \to \mathbb{R}$ with $f_k(0) = 0$, and each f_k vanishing to finite order at 0, such that

$$\left\{ z \in U; r(z) = 2\Re(z_n) + \sum_{k=1}^{n-1} f_k(z_k) < 0 \right\} \simeq \Omega \cap U_\zeta.$$

Let us denote by $\alpha_k(\zeta)$ the order of vanishing of f_k at 0.

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Remark. The finiteness condition here is equivalent to finite type in the case of real analytic pseudo-convex hypersurface $\mathfrak{X} \subset \mathbb{C}^n$ since the Levi form of Ω is diagonalizable (see Kohn [K1, K2]). We also would like to mention that when the Levi form is diagonalizable the finite type is also equivalent to the condition introduced in [K3].

In the following, we will assume that Ω is a bounded domain which is decoupled of finite type near each point of $\partial\Omega$ of non-strict pseudoconvexity. We denote by

$$(\alpha_1, \ldots, \alpha_{n-1}) = \max_{\zeta \in \partial\Omega} (\alpha_1(\zeta), \ldots, \alpha_{n-1}(\zeta)).$$

Let $\zeta \in \partial\Omega$. If ζ is a point of non-strict pseudoconvexity, let f_k be the functions given by Definition 1.1 and let $Z_k = \partial/\partial z_k - (\partial f_k/\partial z_k)(\partial/\partial z_n)$ for $k = 1$ to $n - 1$ and $N = \partial/\partial z_n - \partial/\partial \overline{z_n}$. Then $\{Z_1, \ldots, Z_{n-1}\}$ is a basis of $T^{1,0}(\partial\Omega)$ near ζ and N is the missing direction. As in [Mc1], for x in a neighborhood of ζ and for $0 < \delta \leq 1$, we can construct $n - 1$ functions $\mu_k(x, \delta)$ for $k = 1$ to $n - 1$, giving the size in the Z_k direction of the biggest ball included in Ω. Let us recall some basic properties of the μ_k's (see [NRSW; CNS; C; Mc1]). For every l-tuple of integers $(i_1, \ldots, i_l)$ define smooth functions $\lambda^{(k)}_{i_1,\ldots,i_l}$, $\beta^{(k)}_{i_1,\ldots,i_l}$, and $\gamma^{(k)}_{i_1,\ldots,i_l}$, in a neighborhood of ζ by the equation

$$[X_{i_l}, [\cdots, [X_{i_1}, X_{i_2}] \cdots]] = \lambda^{(k)}_{i_1,\ldots,i_l} N + \beta^{(k)}_{i_1,\ldots,i_l} Z_k + \gamma^{(k)}_{i_1,\ldots,i_l} \overline{Z}_k$$

with $X_{i_j} = Z_k$ or $\overline{Z}_k$ for $j = 1, \ldots, l$.

For each integer $2 \leq l \leq \alpha_k$, we define a smooth function $\Lambda^{(l)}_k$ in a neighborhood of ζ by

$$\Lambda^{(l)}_k(x) = \left[\sum_{j \leq l} |\lambda^{(k)}_{i_1,\ldots,i_j}(x)|\right]^{1/2}.$$

Finally, set

$$\Lambda_k(x, \delta) = \sum_{l=2}^{\alpha_k} \lambda^{(l)}_k(x)\delta^l.$$

The $h \to \mu_k(x, h)$ be the function inverse to $\delta \to \Lambda_k(x, \delta)$. Thus

$$\mu_k(x, h) \approx \min_{2 \leq l \leq \alpha_k} \left(\frac{h}{\Lambda^{(l)}_k(x)}\right)^{1/l}.$$

For every $0 < \delta < 1$, every x in a neighborhood of ζ,

$$c\delta^{1/2} \leq \mu_k(x, \delta) \leq C\delta^{1/\alpha_k}$$

and for every $0 \leq \lambda \leq 1$,

$$\lambda^{1/2}\mu_k(x, \delta) \leq \mu_k(x, \lambda\delta) \leq \lambda^{1/\alpha_k}\mu_k(x, \delta).$$

If $\zeta \in \partial\Omega$ is a point of strict pseudoconvexity, let $\{L_1, \ldots, L_{n-1}\}$ be a basis of holomorphic complex tangential vector fields near ζ. Assume that $\Omega = \{R < 0\}$ and define the following function near ζ:

$$A_2 = \max\{\partial\bar{\partial}R(L_i, \overline{L_j}), \qquad 1 \leq i, j \leq n - 1\}.$$

By strict pseudoconvexity, A_2 is different from zero in a neighborhood of ζ. So, we can define, in a neighborhood of ζ, $\mu_k(x, \delta) = (\delta/A_2(x))^{1/2}$ for $k = 1$ to $n - 1$. Since strictly pseudoconvexity is invariant under biholomorphic change of coordinates, this definition makes sense.

These quantities allow us to define a pseudometric on $\partial\Omega$ as follows. For $x, y \in \partial\Omega$ near ζ,

$$\begin{aligned}\rho(x, y) = \inf\{\delta > 0&: \text{there exists a smooth function } \phi: [0, 1] \to \partial\Omega,\\ \phi(0) = x, \phi(1) &= y, \phi'(t) = \sum_{k=1}^{n-1} (\alpha_k(t)Z_k + \beta_k(t)\overline{Z_k}) + \alpha_n(t)N;\\ |\alpha_k(t)| = |\beta_k(t)| &\leq \mu_k(x, \delta), |\alpha_n(t)| \leq \delta\}.\end{aligned}$$

To obtain a globally defined pseudometric, we use a coordinate patch over $\partial\Omega$. This pseudometric is well defined and by the Campbell–Hausdorff formula, we can show that the corresponding balls $\tilde{B}(x, \delta)$ are equivalent, up to a holomorphic change of coordinates, to a polydisc whose size is δ in the missing direction and $\mu_k(x, \delta)$ in the direction corresponding to Z_k. Furthermore, we have the following additional property on the μ_k's which is that if $x \in \tilde{B}(y, \delta)$ then $\mu_k(x, \delta) \simeq \mu_k(y, \delta)$.

We refer the readers to the celebration article written by Nagel, Stein, and Wainger [NSW] about pseudometrics in a general setting.

This geometry will enable us to estimate the Szegö kernel and to obtain some regularity properties. Some of the results in this paper have been announced in [CG]. We thank our teachers and friends Elias Stein and Aline Bonami for many inspiring conversations.

1. NIS Operators of Order m

As in [RS; NRSW; CNS], we can consider a class of NIS operators. First, let us define the class of bump functions.

Definition 1.1 Let $l \in \mathbb{N}$. A function φ of class $\mathscr{C}^l$ on $\partial\Omega$ is said to be bump of order l if φ is supported on some $\tilde{B}(x_0, \delta)$ and if, for any $\mathbf{i} = (i_1, \ldots, i_{n-1})$ with $|\mathbf{i}| \leq l$,

$$\sup_{z \in \tilde{B}(x_0, \delta)} \prod_{k=1}^{n-1} [\mu_k(x_0, \delta)]^{i_k} |X^{\mathbf{i}} \varphi(z)| \leq 1,$$

where $X^{\mathbf{i}} = X_1^{i_1} \cdots X_{n-1}^{i_{n-1}}$, $X_j = Z_j$ or $\overline{Z}_j$ for $1 \leq j \leq n - 1$.

Now, let us define the class of NIS operators.

Definition 1.2. An operator T

$$T(f)(x) = \int_{\partial\Omega} T(x, y) f(y)\, d\sigma(y)$$

is called a NIS (non-isotropic smoothing) operator of order $\mathbf{m} = (m_1, \ldots, m_{n-1})$ if there exists a family $T_\varepsilon(f)(x) = \int_{\partial\Omega} T_\varepsilon(x, y) f(y)\, d\sigma(y)$ so that

(i) $T_\varepsilon(f) \to T(f)$ in $\mathscr{C}^\infty$ as $\varepsilon \to 0$ whenever $f \in \mathscr{C}^\infty$;

(ii) each $T_\varepsilon(x, y)$ is in $\mathscr{C}^\infty(\partial\Omega \times \partial\Omega)$; and

(iii) the following two properties hold uniformly in ε (we shall omit the subscript ε):

(1)

$$|X_x^{\mathbf{i}} X_y^{\mathbf{j}} T(x, y)| \leq C_{\mathbf{ij}} \frac{\prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k - i_k - j_k}}{V(x, y)},$$

where $\delta = \rho(x, y)$ and $V(x, y) = \delta \cdot \prod_{k=1}^{n-1} \mu_k(x, \delta)^2 \approx \sigma(\tilde{B}(x, \delta))$.

(2) For each l there exists a positive integer N_l such that whenever φ is a bump function of order $\geq N_l$ supported in $\tilde{B}(x_0, \delta)$, then

$$|X^{\mathbf{i}} T(\varphi)(x_0)| \leq C_{\mathbf{i}} \prod_{k=1}^{n-1} \mu_k(x_0, \delta))^{m_k - i_k}$$
$$\times \sup \sum_{|\mathbf{j}| \leq N_l} \prod_{k=1}^{n-1} [\mu_k(x_0, \delta)]^{j_k} |X^{\mathbf{j}}(\varphi)|(x_0),$$

where $|\mathbf{i}| = l$.

(iv) The same estimates hold for the operator T^*, i.e., for the operator of kernel $\overline{T}(y, x)$.

In the following, the order of the bump functions will be allowed to change from time to time and it will be assumed to be large enough. Now let us discuss some properties of NIS operators. First, we are interested in the answers of the following three questions.

QUESTION 1. *Let K be an* NIS *operator of order* $\mathbf{m}_1$ *and let L be an* NIS *operator of order* $\mathbf{m}_2$. *What can we say about KL?*

We are going to show that KL is an operator of order $\mathbf{m}_1 + \mathbf{m}_2$, under the assumption that

$$\sum_{m_k^{(2)}+m_k^{(1)}\geq 0} \frac{m_k^{(2)} + m_k^{(1)}}{2} + \sum_{m_k^{(2)}+m_k^{(1)}<0} \frac{m_k^{(2)} + m_k^{(1)}}{\alpha_k} < 1 + \sum_{k=1}^{n-1} \frac{2}{\alpha_k}.$$

Let $T = KL$ such that

$$T(x, y) = \int_{\partial\Omega} K(x, z)L(z, y)\, d\sigma(z).$$

Obviously T satisfies properties (i) and (ii). Let us look at the third property. We shall drop the subscript in the following. First, we want to estimate $|T(x, y)|$. Let $\delta = \rho(x, y)$ and let φ_1 be some bump function supported in $\tilde{B}(x, \delta/2)$ and $\varphi_1(z) \equiv 1$ in $\tilde{B}(x, c\delta)$ for some $c < \frac{1}{4}$. We also assume that φ_2 is a bump function supported in $\tilde{B}(y, \delta/2)$ and $\varphi_2(z) \equiv 1$ in $\tilde{B}(x, c\delta)$. Define

$$\Psi = 1 - \varphi_1 - \varphi_2.$$

Now we have

$$\begin{aligned} \int_{\partial\Omega} K(x, z)L(z, y)\, d\sigma(z) &= \int_{\partial\Omega} K(x, z)L(z, y)\varphi_1(z)\, d\sigma(z) \\ &\quad + \int_{\partial\Omega} K(x, z)L(z, y)\varphi_2(z)\, d\sigma(z) \\ &\quad + \int_{\partial\Omega} K(x, z)L(z, y)\Psi(z)\, d\sigma(z) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Let us estimate I. Since L is order $\mathbf{m}_2$, we have that

$$|L(z, y)| \leq C \frac{\prod_{k=1}^{n-1} \mu_k(z, \rho(z, y))^{m_k^{(2)}}}{V(z, y)}.$$

Here $\rho(z, x) \le \delta/2$ and hence $\rho(y, z) \approx \delta$ and $V(z, y) \approx V(x, y)$, the volume of the ball $\tilde{B}(y, \delta)$. It follows that

$$\begin{aligned}|\mathrm{I}| &\le C \cdot \frac{\prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k^{(2)}}}{V(x, y)} |K(\varphi_1)(x)| \\ &\le C \cdot \frac{\prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k^{(2)} + m_k^{(1)}}}{V(x, y)},\end{aligned}$$

since, by assumption, if φ_1 is a bump function of order sufficiently large, $|K(\varphi_1)(x)| \le C \prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k^{(1)}}$. We handle term II similarly. Now let us look at term III:

$$\begin{aligned}|\mathrm{III}| &= \left| \int_{\partial\Omega} K(x, z) L(z, y) \Psi(z) \, d\sigma(z) \right| \\ &\le \int_{\rho(x,z) \ge c\delta, \rho(y,z) \ge c\delta} |K(x, z)| \cdot |L(z, y)| \, d\sigma(z) \\ &\le \int_{\rho(x,z) \le 2\delta} + \int_{\rho(x,z) > 2\delta} \\ &= \mathrm{III}_1 + \mathrm{III}_2.\end{aligned}$$

In the region of the integral III_1, we have $\rho(x, y) \approx \rho(x, z)$ and $V(x, y) \approx V(x, z)$. Similarly, $\rho(x, y) \approx \rho(y, z)$. Therefore,

$$\begin{aligned}\mathrm{III}_1 &\le C \cdot \frac{\prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k^{(2)} + m_k^{(1)}}}{V^2(x, y)} \int_{\rho(x,z) \le 2\delta} d\sigma(z) \\ &\le C \cdot \frac{\prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k^{(2)} + m_k^{(1)}}}{V^2(x, y)} \sigma(\tilde{B}(x, 2\delta)) \\ &\le C \cdot \frac{\prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k^{(2)} + m_k^{(1)}}}{V(x, y)}.\end{aligned}$$

In the region of the integral III_2, we have $\rho(z, y) \approx \rho(z, x)$ and, hence, $V(z, y) \approx V(z, x)$. Furthermore, since $z \in \tilde{B}(x, c\rho(z, x))$, $\mu_k(z, \rho(z, y)) \approx \mu_k(x, \rho(z, y))$. Then we have

$$\begin{aligned}\mathrm{III}_2 &\le C \cdot \int_{\rho(x,z) > 2\delta} \frac{\prod_{k=1}^{n-1} \mu_k(x, \rho(z, x))^{m_k^{(2)} + m_k^{(1)}}}{V^2(x, y)} d\sigma(z) \\ &\le C \cdot \sum_{l=1}^{\infty} \int_{2^l\delta \le \rho(x,z) \le 2^{l+1}\delta} \frac{\prod_{k=1}^{n-1} \mu_k(x, \rho(z, x))^{m_k^{(2)} + m_k^{(1)}}}{V^2(x, y)} d\sigma(z)\end{aligned}$$

$$\leq C \cdot \sum_{l=1}^{\infty} \prod_{k=1}^{n-1} \mu_k(x, 2^l\delta)^{m_k^{(2)} + m_k^{(1)}} (\sigma(B(x, 2^l\delta))^{-1}$$

$$\leq C \cdot \frac{\prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k^{(2)} + m_k^{(1)}}}{\sigma(B(x, \delta))}$$

$$\times \sum_{l=1}^{\infty} \left(\prod_{m_k^{(2)} + m_k^{(1)} \geq 0} (2^{l/2})^{m_k^{(2)} + m_k^{(1)}} \prod_{m_k^{(2)} + m_k^{(1)} < 0} (2^l)^{m_k^{(2)} + m_k^{(1)}/\alpha_k} \cdot 2^{-l} \cdot \prod_{k=1}^{n-1} 2^{-2l/\alpha_k} \right)$$

$$\leq C \cdot \frac{\prod_{k=1}^{n-1} \mu_k(x, \delta)^{m_k^{(2)} + m_k^{(1)}}}{\sigma(B(x, \delta))}$$

$$\times \sum_{l=1}^{\infty} 2^{l(\Sigma_{m_k^{(2)} + m_k^{(1)} \geq 0}(m_k^{(2)} + m_k^{(1)})/2 + \Sigma_{m_k^{(2)} + m_k^{(1)} < 0}(m_k^{(2)} + m_k^{(1)}/\alpha_k - 1 - \Sigma_{k=1}^{n-1} 2/\alpha_k)}.$$

So it converges if

$$\sum_{m_k^{(2)} + m_k^{(1)} \geq 0} \frac{m_k^{(2)} + m_k^{(1)}}{2} + \sum_{m_k^{(2)} + m_k^{(1)} < 0} \frac{m_k^{(2)} + m_k^{(1)}}{2} < 1 + \sum_{k=1}^{n-1} \frac{2}{\alpha_k}.$$

To finish the proof, we have just to remark that the same kind of estimates that holds for the derivatives of T, holds for T^* and its derivatives.

The following lemma is useful and generalizes the kind of computation made just before.

LEMMA 1.1. *For some positive numbers β, β_k*

$$\int_{\rho(x,z) > \delta} \prod_{k=1}^{n-1} \mu_k^{\beta_k}(x, \rho(x, z)) \times V(x, z)^{-(1+\beta)} \, d\sigma(z) \leq \prod_{k=1}^{n-1} \mu_k^{\beta_k}(x, \delta) \times \sigma(\tilde{B}(x, \delta))^{-\beta}$$

if $\sum_{k=1}^{n-1} \beta_k < 2(1 + \sum_{k=1}^{n-1} 2/\alpha_k)\beta$;

$$\int_{\rho(x,z) \leq \delta} \prod_{k=1}^{n-1} \mu_k^{\beta_k}(x, \rho(x, z)) \times V(x, z)^{-(1+\beta)} \, d\sigma(z) \leq \prod_{k=1}^{n-1} \mu_k^{\beta_k}(x, \delta) \times \sigma(\tilde{B}(x, \delta))^{-\beta}$$

if $\sum_{k=1}^{n-1} \beta_k/\alpha_k > n\beta$.

QUESTION 2. *If T is an operator of order $\mathbf{m}$ with $m_k \geq 0$ for $k = 1$ to $n - 1$ (we will say that T is an operator of positive order), when can we say that $X_i T$ is again an operator of positive order?*

First, if $m_i \geq 1$, then, obviously, $X_i T$ is an operator of positive order

$$(m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_{n-1}).$$

Otherwise, if we assume that $\sum_{k \neq i} 2/\alpha_k \cdot m_k + m_i \geq 1$, then we can show that $X_i T$ is of positive order $(m_1 - \nu_1, \ldots, m_j - \nu_j, \ldots, m_{n-1} - \nu_{n-1})$ for any choice of $\{\nu_j\}$ satisfying $0 \leq \nu_j \leq m_j$ for $j = 1$ to $n - 1$, and $\sum_{k \neq i} 2/\alpha_k \cdot \nu_k + \nu_i > 1$. To see this, it suffices to use the basic properties of the μ_k's. In fact, we can show that an operator of order $\mathbf{m}$ is also an operator of order $\mathbf{l}$ as long as

$$\sum_{m_k \geq l_k} \frac{m_k - l_k}{\alpha_k} \geq \sum_{m_k < l_k} \frac{l_k - m_k}{2}.$$

Now we come to the third question.

QUESTION 3. *Are there non-trivial NIS operators of order* $\mathbf{m} \neq \mathbf{0}$?

First of all, we should point out that the pseudometric associated to the Szegö projection for strongly pseudoconvex domains in $\mathbb{C}^n$ or finite type domains in $\mathbb{C}^2$ is the same as the pseudometric associated to the sum of square of vector fields and the operator $\square_b$. If K is the parametrix of $\square_b$, it follows that K is an NIS operator of order 2 and $X_1 K$ is an NIS operator of order 1. Unlike the cases of strong pseudoconvexity and finite type in $\mathbb{C}^2$, the parametrix G of $\square_b$ for decoupled domains of finite type in $\mathbb{C}^3$ is controlled by the above pseudometric ρ and another non-equivalent metric $\tilde{\rho}$. For example, let us consider the domain $\{(z_1, z_2, z_3)\colon \Im z_3 > \mathcal{P}(z_1) + |z_2|^2\}$ in $\mathbb{C}^3$ with $\mathcal{P}$ a subharmonic, but not a harmonic, polynomial; then

$$|G(x, y)| \leq C \frac{\tilde{\rho}^2(x, y)}{\sigma(B^{\tilde{\rho}}(y, \tilde{\rho}(x, y)))} \log\left(2 + \frac{\tilde{\rho}(x, y)}{\rho(x, y)}\right)$$

and the log cannot be removed. Here

ρ = the metric associated to $\sum_{j=1}^{2} (X_j^2 + Y_j^2)$ and B^ρ is the associated ball;

$\tilde{\rho}$ = the metric associated to $(1 - \lambda)^{-1} (X_1^2 + Y_1^2 + \lambda(X_2^2 + Y_2^2))$ and $B^{\tilde{\rho}}$ is the associated ball;

where $\lambda = \frac{1}{4} \Delta \mathcal{P}$. However, according to a theorem of Machedron [M], we know that

$$|Z^J \overline{L}_{2x} G(x, y)| \leq C \frac{\rho^{1-|J|}(x, y)}{\sigma(B^\rho(y, \rho(x, y)))}$$

for Z^J a $|J|$th-order monomial in X_j, Y_j, $j = 1, 2$, taken in x, y. This tells us $\bar{L}_{2x}G(x, y)$ is an NIS operator of order $\mathbf{m}$ with $|\mathbf{m}| = 1$.

2. Some Regularity Properties of the NIS Operators of Order 0

Theorem 2.1. *Suppose T is an operator of order 0. Then T maps $L^p(\partial\Omega)$ to $L^p(\partial\Omega)$ for $1 < p < \infty$.*

Proof. We have to use David–Journé theorem (Theorem T(1)) (see [DJ; NRSW] for a use of this theorem in this context). By assumption on T, we know that $T(1) \in \mathscr{C}^\infty$ and hence $T(1)$ belongs to BMO. We have to verify that

$$\|T(\varphi)\|_{L^2(\partial\Omega)} \leq C \cdot (\sigma(\tilde{B}(x_0, \delta)))^{1/2},$$

where φ is a bump function associated to the ball $\tilde{B}(x_0, \delta)$. We know that

$$|T(\varphi)(x)| \leq C$$

for all $x \in \tilde{B}(x_0, 2\delta)$. On the other hand, for any $x \notin \tilde{B}(x_0, 2\delta)$

$$\begin{aligned} |T(\varphi(x))| &= \left| \int_{\partial\Omega} T(x, y)\varphi(y)\, d\sigma(y) \right| \\ &\leq C \int_{\tilde{B}(x_0, \delta)} \frac{1}{V(x, y)}\, d\sigma(y) \\ &\leq \sigma(\tilde{B}(x_0, \delta)) \cdot V(x, x_0)^{-1}. \end{aligned}$$

So, we obtain

$$\begin{aligned} \|T(\varphi)\|^2_{L^2(\partial\Omega)} &\leq C\sigma(\tilde{B}(x_0, 2\delta)) + \sigma(\tilde{B}(x_0, \delta))^2 \int_{\rho(x, x_0) \geq 2\delta} V(x, x_0)^{-2}\, d\sigma(x) \\ &\leq C\sigma(\tilde{B}(x_0, \delta)). \quad \blacksquare \end{aligned}$$

Now, let $\mathscr{H}^p(\partial\Omega)$ be the Hardy space defined as in [CK] (roughly speaking, it is a space of distributions on $\partial\Omega$ which have some maxmimal function in $L^p(\partial\Omega)$). By the result of [MS; U], we know that there exists p_0 such that, for $p_0 < p \leq 1$, each function in $\mathscr{H}^p(\partial\Omega)$ has an atomic decomposition. Using the same method as in [CK], we obtain the following theorem.

Theorem 2.2. *Let T be any NIS operator of order 0. Then T maps $\mathscr{H}^p(\partial\Omega)$ to $\mathscr{H}^p(\partial\Omega)$ boundedly for $p_0 < p \leq 1$.*

Now let us turn to the Hölder estimates. For $0 < \alpha < \min_{k=1,\ldots,n-1} (1/\alpha_k)$, we define

$$\Gamma_\alpha(\partial\Omega) = \{f \in L^\infty(\partial\Omega) : |f(x) - f(y)| \le C\rho^\alpha(x, y), \forall x, y \in \partial\Omega\}.$$

We want to show that any NIS operator of order 0 maps $\Gamma_\alpha(\partial\Omega)$ into itself for small α. Indeed, we have the following theorem.

THEOREM 2.3. *Let T be any operator of order* 0 *then T maps $\Gamma_\alpha(\partial\Omega)$ to itself for $0 < \alpha < \min_{k=1,\ldots,n-1} (1/\alpha_k)$.*

Proof. Let $x_1, x_2 \in \partial\Omega$, $\rho(x_1, x_2) = \delta$, and φ be a bump function which is equal to 1 on $B(x_1, 2\delta)$ and vanishes outside the ball $B(x_1, c\delta)$, $c > 2$. Let us denote by $F(x) = Tf(x)$ for $f \in \Gamma_\alpha(\partial\Omega)$. Then we have

$$\begin{aligned}
F(x_1) - F(x_2) &= \int T(x_1, y)(f(y) - f(x_1))\varphi(y)\, d\sigma(y) \\
&\quad - \int T(x_2, y)(f(y) - f(x_1))\varphi(y)\, d\sigma(y) \\
&\quad + \int (T(x_1, y) - T(x_2, y))(f(y) - f(x_1))(1 - \varphi(y))\, d\sigma(y) \\
&\quad + f(x_1)(T(1)(x_1) - T(1)(x_2)) = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

The first term

$$\begin{aligned}
|\mathrm{I}| &\le \left| \int T(x_1, y)(f(y) - f(x_1))\varphi(y)\, d\sigma(y)\right| \\
&\le \int_{\rho(x_1, y) \le c\delta} \frac{\rho^2(x_1, y)}{V(x_1, y)}\, d\sigma(y) \\
&\le \sum_{k=0}^{\infty} \int_{c2^{-(k+1)}\delta \le \rho(x_1, y) \le c2^{-k}\delta} \frac{\rho^2(x_1, y)}{V(x_1, y)}\, d\sigma(y) \\
&= \sum_{k=0}^{\infty} (2^{-k}\delta)^\alpha \times \frac{\sigma(B(x_1, c2^{-k}\delta))}{\sigma(B(x_1, c'2^{-k}\delta))} \le C\delta^\alpha.
\end{aligned}$$

The second term

$$\begin{aligned}
|\mathrm{II}| &\le \left| \int T(x_2, y)(f(y) - f(x_1))\varphi(y)\, d\sigma(y)\right| \\
&\le \left| \int T(x_2, y)(f(y) - f(x_2))\varphi(y)\, d\sigma(y)\right| \\
&\quad + |f(x_2) - f(x_1)| \cdot |T(\varphi)(x_2)| \le C\delta^\alpha,
\end{aligned}$$

since $|T(\varphi)(x_2)|$ is bounded (the first part is of the same kind than I). The third term

$$\begin{aligned}|\mathrm{III}| &= |\textstyle\int (T(x_1, y) - T(x_2, y))(f(y) - f(x_1))(1 - \varphi(y))\, d\sigma(y)| \\ &\leq \delta \cdot \int_{\rho(x_1, y) \geq c\delta} \frac{\rho^{\alpha-1}(x_1, y)}{V(x_1, y)}\, d\sigma(y) \\ &\quad + \sum_{k=1}^{n-1} \mu_k(x_1, \delta) \int_{\rho(x_1, y) \geq c\delta} \frac{\mu_k(x_1, \rho(x_1, y))^{-1} \rho^{\alpha}(x_1, y)}{V(x_1, y)}\, d\sigma(y) \\ &\leq C\delta\delta^{\alpha-1} + \mathrm{III}',\end{aligned}$$

by the Taylor formula, if $\alpha < 1$. Now we look at the term III′,

$$\begin{aligned}\mathrm{III}' &\leq \sum_{k=1}^{n-1} \mu_k(x_1, \delta) \sum_{j=0}^{\infty} \int_{c2^j\delta \leq \rho(x_1, y) \leq c2^{j+1}\delta} \frac{\mu_k(x_1, \rho(x_1, y))^{-1} \rho^{\alpha}(x_1, y)}{V(x_1, y)}\, d\sigma(y) \\ &\leq C \sum_{k=1}^{n-1} \mu_k(x_1, \delta) \cdot \sum_{j=0}^{\infty} (2^j\delta) 2^{-j/\alpha_k} \cdot \mu_k(x_1, \delta)^{-1} \leq C\delta^{\alpha}\end{aligned}$$

if $\alpha - 1/\alpha_k < 0$, for all k, i.e., if $0 < \alpha < \min\limits_{k=1,\ldots,n-1} (1/\alpha_k)$. Now let us turn to the fourth term. By assumption, $T(1)$ is a smooth function, so

$$\mathrm{IV} \leq C\|f\|_{\infty}|x_1 - x_2| \leq \delta^{\alpha}$$

since, by assumption $\alpha < \min\limits_{k=1,\ldots,n-1} (1/\alpha_k)$ and, by definition,

$$\begin{aligned}\rho(x_1, x_2) &\simeq \sum_k (\Lambda_k(x_1, |\langle x_1 - x_2, Z_k \rangle|) \\ &\quad + \Lambda_k(x_1, |\langle x_1 - x_2, \overline{Z}_k \rangle|) + |\langle x_1 - x_2, N \rangle| \\ &\geq c|x_1 - x_2|^{\alpha_k} \qquad \text{for any } k = 1 \text{ to } n - 1. \quad \blacksquare\end{aligned}$$

3. Szegö Projection

Fix a smooth vector field $\mathbf{n}$ defined near $\partial\Omega$ which is tranverse to the boundary. For any $x \in \partial\Omega$, let $t \to \mathbf{n}(x, t)$ denote the integral curve of $\mathbf{n}$. Fix a smooth function a on $\overline{\Omega}$ and define the kernel $C(x, y)$ on $\partial\Omega \times \partial\Omega$ by

$$C(x, y) = \int_0^c a(\mathbf{n}(x, t)) B(\mathbf{n}(x, t), y)\, dt,$$

where $B(z, y)$ is the Bergman kernel. We also write

$$C_\varepsilon(x, y) = \int_\varepsilon^c a(\mathbf{n}(x, t))B(\mathbf{n}(x, t), y)\, dt$$

and note that $C_\varepsilon \in \mathscr{C}^\infty(\partial\Omega \times \partial\Omega)$.

First, we are going to prove that C is an operator of order 0. In [Mc2], it is proved that, for $z, w \in \Omega$, if

$$\theta \approx \rho(\pi(z), \pi(w)) + |r(z)| + |r(w)|,$$

then

$$|B_{\overline{\Omega}}(z, w)| \leq C\theta^{-2} \prod_{k=1}^{n-1} \mu_k^{-2}(\pi(z), \theta).$$

So we have

$$|B(\mathbf{n}(x, t), y)| \leq C\delta^{-2} \prod_{k=1}^{n-1} \mu_k^{-2}(x, \delta)$$

and

$$|B(\mathbf{n}(x, t), y)| \leq Ct^{-2} \prod_{k=1}^{n-1} \mu_k^{-2}(x, \delta), \tag{3.1}$$

since $\theta \gtrsim t$ and $\theta \gtrsim \rho(x, y) = \delta$. It follows that

$$\begin{aligned} |C(x, y)| &\leq \int_0^\delta a(\mathbf{n}(x, t))B(\mathbf{n}(x, t), y)\, dt + \int_\delta^c a(\mathbf{n}(x, t))B(\mathbf{n}(x, t), y)\, dt \\ &\leq C\delta^{-1} \prod_{k=1}^{n-1} \mu_k^{-2}(\mathrm{x}, \delta). \end{aligned}$$

We also know that

$$|X_x^{\mathbf{i}} X_y^{\mathbf{j}} B(\mathbf{n}(x, t), y)| \leq C_{\mathbf{ij}}\theta^{-2} \prod_{k=1}^{n-1} \mu^{-2-i_k-j_k}(x, \theta).$$

This allows us to see that

$$|X_x^{\mathbf{i}} X_y^{\mathbf{j}} C(x, y)| \leq C_{\mathbf{ij}} \frac{\prod_{k=1}^{n-1} \mu_k^{-i_k-j_k}(x, \delta)}{\delta \prod_{k=1}^{n-1} \mu_k^2(x, \delta)}.$$

Now, we want to estimate $|X^{\mathbf{i}}C(\varphi)(x_0)|$ for some bump function φ. First, let us take $|\mathbf{i}| = 0$. We write

$$\begin{aligned} |C(\varphi)(x_0)| &= \left| \int_{\partial\Omega} C(x_0, y)\varphi(y)\, d\sigma(y) \right| \\ &= \int_0^\delta (\cdot)\, dt + \int_\delta^c (\cdot)\, dt \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Since $|\nabla r| = 1$, where $r = 0$, we have

$$\begin{aligned} \int_{\partial\Omega} B(z, y)\varphi(y)\, d\sigma(y) &= \sum_{k=1}^{n} \int_\Omega \frac{\partial}{\partial w_k} (B(z, w)\bar{r}_k(w)\tilde{\varphi}(w))\, dw \\ &= B(\Delta r\tilde{\varphi})(z) + \sum_{k=1}^{n} B\left(\bar{r}_k \frac{\partial\tilde{\varphi}}{\partial w_k}\right)(z), \end{aligned}$$

where $\bar{r}_k(w) = \partial r_k(w)/\partial\bar{w}_k$ and $\tilde{\varphi}(w)$ is the extension of φ with support in

$$P(x_0, \delta) = \{z;\ \pi(z) \in B(x_0, \delta),\ 0 \le -r(z) \le \delta\}$$

(explictly, we set $\tilde{\varphi}(z) = \phi(\pi(z))\chi(-r(z)/\delta)$, where χ is a $\mathscr{C}^\infty$ function with $\chi(t) = 1$ for $0 \le t \le \frac{1}{2}$ and $\chi(t) = 0$ when $t \ge 1$). By the same method used in [NRSW], we can show that for any bump fraction $\tilde{\varphi}$ in $P(p, \delta)$, of order $\ge N_l$,

$$\begin{aligned} \sup_{z\in P(p,\delta)} |X^{\mathbf{i}}B(\tilde{\varphi})(z)| &\le C \cdot \prod_{k=1}^{n-1} \mu_k(p, \delta)^{-i_k} \\ &\times \sum_{|\mathbf{j}|+\beta+\gamma\le N_l} \sup_{z\in P(p,\delta)} \delta^{\beta+\gamma} \prod_{k=1}^{n-1} \mu_k(p, \delta)^{j_k} \left| X^{\mathbf{j}} \left(\frac{\partial}{\partial z_n}\right)^\beta \left(\frac{\partial}{\partial \bar{z}_n}\right)^\gamma (\tilde{\varphi})(z) \right|. \end{aligned}$$

This gives

$$\left| \int_{\partial\Omega} B(z, y)\varphi(y)\, d\sigma(y) \right| \le C\delta^{-1}$$

for any φ bump of order sufficiently large.

It follows that

$$|I| \le C\delta^{-1} \cdot \delta \le C.$$

Now, we estimate II using (3.1) and the fact that φ is supported in $\tilde{B}(x_0, \delta)$ which has volume $\le C\delta \prod_{k=1}^{n-1} \mu_k^2(x_0, \delta)$. This gives

$$|\mathrm{II}| \le C\delta \prod_{k=1}^{n-1} \mu_k^2(x_0, \delta) \int_\delta^c t^{-2} \prod_{k=1}^{n-1} \mu_k^{-2}(x_0, \delta)\, dt$$
$$\le C.$$

We also have

$$\begin{aligned}
|X^{\mathbf{i}}C(\varphi)(x_0)| &= \left| X^{\mathbf{i}}_{x_0} \int_{\partial\Omega} C(x_0, y)\varphi(y)\, d\sigma(y) \right| \\
&= \left| X^{\mathbf{i}}_{x_0} \int_{\partial\Omega} \int_0^c a(\mathbf{n}(x_0, t)) B(\mathbf{n}(x_0, t), y)\varphi(y)\, dt d\sigma(y) \right| \\
&\le \left| \int_0^\delta X^{\mathbf{i}}_{x_0} \int_{\partial\Omega} B(\mathbf{n}(x_0, t), y)\varphi(y)\, d\sigma(y) a(\mathbf{n}(x_0, t))\, dt \right| \\
&\quad + \left| \int_\delta^c X^{\mathbf{i}}_{x_0} \int_{\partial\Omega} B(\mathbf{n}(x_0, t), y)\varphi(y)\, d\sigma(y) a(\mathbf{n}(x_0, t))\, dt \right|.
\end{aligned}$$

We do as before and the main term gives

$$\begin{aligned}
&\delta^{-1} \cdot \delta \cdot \prod_{k=1}^{n-1} \mu_k^{-i_k}(x_0, \delta) + \frac{C \cdot \prod_{k=1}^{n-1} \mu_k^{-i_k}(x_0, \delta) \left(\delta \cdot \prod_{k=1}^{n-1} \mu_k^2(x_0, \delta) \right)}{\delta \cdot \prod_{k=1}^{n-1} \mu_k^2(x_0, \delta)} \\
&\lesssim \prod_{k=1}^{n-1} \mu_k^{-i_k}(x_0, \delta).
\end{aligned}$$

So $|X^{\mathbf{i}}C(\varphi)(x_0)| \le \prod_{k=1}^{n-1} \mu_k^{-i_k}(x_0, \delta)$. Finally, let us consider the action of C^* on φ. As noted above, we can write $w \in \Omega$ as $w = \mathbf{n}(y, t)$ for w near $\partial\Omega$. We also have that $\tilde{\varphi}(w) = \varphi(y)$ whenever $0 \le t \le c\delta$. Also, if we make the change of variables near $\partial\Omega$ given by $w \leftrightarrow (y, t)$, then $d\sigma(y)\, dt = J(w)\, dw$, where J is smooth near $\partial\Omega$. Thus

$$\begin{aligned}
C^*(\varphi)(x_0) &= \int_{\partial\Omega} \int_0^c B(x_0, \mathbf{n}(y, t)) \bar{a}(\mathbf{n}(y, t))\varphi(y)\, dt d\sigma(y) \\
&= B(\tilde{\varphi}\bar{a}J)(x_0) + \mathbb{O}\left(\int \int_{\substack{y \in P(x_0, \delta) \\ \delta \ge t \ge c\delta}} |B(x_0, \mathbf{n}(y, t))|\, dt\, d\sigma(y) \right).
\end{aligned}$$

Now, $\tilde{\varphi}\bar{a}J$ satisfies the same kind of estimates as $\tilde{\varphi}$ does; hence we have $|B(\tilde{\varphi}\bar{a}J)(x_0)| \le c$. However,

$$|B(x_0, \mathbf{n}(y, t))| \le c\delta^{-2} \prod_{k=1}^{n-1} \mu_k(x_0, \delta)^{-2}$$

whenever $y \in P(x_0, \delta)$ and $t = \delta$; hence the $\mathcal{O}$ term is also bounded by a constant. The same kind of estimates holds for the derivatives and hence allows to conclude that C is an operator of order 0. Now, we are going to show the following result.

PROPOSITION 3.1. *There exist an operator A of order* 0 *and E an operator which behaves as $\sum_{j=1}^{n-1} A_j$, where A_j is an operator of order* $(0, \ldots, 0, \overset{j}{\check{1}}, 0, \ldots, 0)$ *so that*

$$S = A + E \cdot S.$$

Suppose that $f \in \mathscr{C}^\infty(\partial\Omega)$ and $F = S(f)$ so that F is holomorphic in Ω. We can find $a_j(z)$ smooth near $\partial\Omega$ so that $\mathbf{n}(F) = \sum_{j=1}^{n} a_j(z)\partial F/\partial z_j$. Next, we denote by $x' = \mathbf{n}(x, c)$. Then since

$$\partial F/\partial z_j(z) = \int_\Omega B(z, w)\partial F/\partial z_j(w)\, dw,$$

we have, when $x \in \partial\Omega$,

$$\begin{aligned} F(x') - F(x) &= \sum_{j=1}^{n} \int_0^c \left\{ \int_\Omega a_j(\mathbf{n}(x, t)) B(\mathbf{n}(x, t), w) \frac{\partial F}{\partial z_j}(w)\, dw \right\} dt \\ &= \sum_{j=1}^{n} \int_{\partial\Omega} \int_0^c a_j(\mathbf{n}(x, t)) B(\mathbf{n}(x, t), y) r_j(y) F(y)\, dt\, d\sigma(y) \end{aligned}$$

by Stokes formula, where $r_j = \partial r/\partial z_i$. Hence

$$F(x') - F(x) = \sum_{j=1}^{n} C_j(r_j F)(x),$$

where C_j is the operator of order 0 with the kernel constructed with the smooth function a_j. However, $F(x') = \int_{\partial\Omega} P(x', y)F(y)\, d\sigma(y) = P'(F)(x)$, where $P(x', y)$ is the Poisson kernel for Ω; since x' lies strictly in the interior of Ω, while the mapping $x \to x'$ is smooth, it therefore follows that the kernel P is $\mathscr{C}^\infty$ in both variables, and so, in particular this operator behaves as $\sum_{j=1}^{n-1} A_j$, where A_j is an operator of order $(0, \ldots, \overset{j}{\check{1}}, \ldots, 0)$. So, we have that

$$S(f) = P'(S(f)) - \sum_{j=1}^{n} C_j(r_j S(f)).$$

However, $C_j(f - S(f)) = 0$, since the kernel of C_j is conjugate holomorphic in the second variable. Thus

$$\begin{aligned} S(f) &= P'(S(f)) - \sum C_j(r_j S(f)) - r_j C_j(f - S(f)) \\ &= A(f) + ES(f), \end{aligned}$$

where

$$\begin{aligned} A &= -\sum r_j C_j \\ E &= \sum (r_j C_j - C_j r_j) + P'. \end{aligned}$$

The proposition is proved if we remark that $C_j r_j - r_j C_j$ behaves as $\sum_{j=1}^{n-1} A_j$, where A_j is an operator of order $(0, \ldots, 0, \check{1}, 0, \ldots, 0)$.

THEOREM 3.2. *The Szegö kernel is an operator of order* 0.

The proof follows the same lines as in [NRSW]. We have that $S = A + ES$; then, by iteration,

$$S = \sum_{j=0}^{n-1} E^j A + E^N S.$$

Taking the adjoints (since $S^* = S$) we have also that

$$S = \sum_{j=0}^{n-1} A^*(E^*)^j + S(E^*)^N.$$

If we substitute in the above, we get

$$S = \sum_{j=0}^{n-1} E^j A + E^N \sum_{j=0}^{n-1} A^*(E^*)^j + E^N S(E^*)^N,$$

so

$$S = S_N + E^N S(E^*)^N,$$

where S_N is an operator of order 0 for each N. Therefore it suffices to prove that for each fixed $\mathbf{I}$ and $\mathbf{J}$, if $K_N(x, y)$ is the kernel of $E^N S(E^*)^N$, then we can choose N so large that $X_x^{\mathbf{I}} X_j^{\mathbf{J}} K_N(x, y)$ is bounded. However, this quality is the kernel of the operator $X^{\mathbf{I}} E^N S(E^*)^N (X^{\mathbf{J}})^*$. So it suffices to show that this operator maps $L^1(\partial\Omega)$ to $L^\infty(\partial\Omega)$ for N sufficiently large. Since S maps $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$, it suffices to show that, for N large enough, $(E^*)^N(X^{\mathbf{J}})^*$ maps $L^1(\partial\Omega)$ to $L^2(\partial\Omega)$ and that $X^{\mathbf{I}}E^N$ maps $L^2(\partial\Omega)$ to $L^\infty(\partial\Omega)$.

First, for L large enough, we can assume that $X^{\mathbf{I}}E^L$ is an operator of order 0. Then, for $N \geq L$, $X^{\mathbf{I}}E^N$ can be written as a product of $N - L$ operators which behave as $\sum_j A_j$. The same holds for $(E^*)^N(X^{\mathbf{J}})^*$. Let A be such an operator. By Lemma 1.1, we can see that

$$\int_{\partial\Omega} |A(x, y)|^r \, d\sigma(y) \leq C$$

independently of x as long as $1 < r \leq n\alpha_k/(n\alpha_k - 1)$ for every $k = 1$ to $n - 1$. In fact, we have

$$\int_{\rho(x,y)<\delta} |A(x, y)|^r \, d\sigma(y) \leq C \int_{\rho(x,y)<\delta} \left(\frac{\sum_k \mu_k(x, \rho(x, y))}{V(x, y)}\right)^r d\sigma(y)$$

$$\leq \sum_k \left(\frac{\mu_k(x, \delta)}{\sigma(\tilde{B}(x, \delta))^{1/r'}}\right)^r$$

by Lemma 1.1, since $r/\alpha_k > n(r - 1)$ by assumption,

$$\leq C \sum_k \left(\frac{\delta^{1/\alpha_k}}{\delta^{n/r'}}\right)^r$$

$$\leq C \qquad \text{since } \frac{1}{r'} < \frac{1}{n\alpha_k} \text{ for } k = 1 \text{ to } n - 1.$$

Hence, such an operator maps $L^1(\partial\Omega)$ to $L^r(\partial\Omega)$ and $L^{r'}(\partial\Omega)$ to $L^\infty(\partial\Omega)$. By interpolation, we conclude that such an operator maps $L^p(\partial\Omega)$ to $L^q(\partial\Omega)$ if $1/q = 1/p - 1/r'$. Hence if N is sufficiently large, iterating this we obtain the result.

As a corollary, we obtain that the Szegö projection is bounded from $L^p(\partial\Omega)$ to itself for any $1 < p < \infty$. In fact, we can prove better, thanks to a particular commutation property (Lemma 3.4) satisfied by S and any first-order differential operator. Denote by $L^p_k(\partial\Omega)$ the Sobolev space of order k. Then, we have the following theorem.

THEOREM 3.3. *The Szegö operator maps $L^p_k(\partial\Omega)$ to itself for $1 < p < \infty$ and $k \in \mathbb{N}$.*

This theorem follows from the following special commutation property which appears in $\mathbb{C}^2$ in [CNS].

LEMMA 3.4. *Let Y be any first-order differential operator. There exists NIS operators $A_1, \ldots, A_{2n-1}$ of smoothing order strictly bigger than 0, a*

NIS operator of order 0, *differential operators* $Y_1, \ldots, Y_{2n-1}$ *of order* 1 *and an operator E of smoothing order arbitrarily larger so that*

$$[Y, S] = \sum_{j=1}^{2n-1} A_j Y_j + A_0 + E.$$

By the same method, we obtain the corresponding regularity in $\mathscr{H}^p$ spaces. Let $\mathscr{H}^p(\Omega)$ be the holomorphic Hardy space in Ω; i.e., it is the set of holomorphic functions g satisfying

$$\sup_{0<\varepsilon<1} \int_{\partial\Omega_\varepsilon} |g(\zeta)|^p \, d\sigma_\varepsilon(\zeta) < \infty,$$

where $\Omega_\varepsilon = \{z \in \Omega; r(z) < -\varepsilon\}$. We denote by $\mathscr{H}^p_k(\Omega)$ the set of holomorphic functions g such that each component of $\nabla^j g$ belongs to $\mathscr{H}^p(\Omega)$ for any integer $j \leq k$. Also, we denote by $\mathscr{H}^p_k(\partial\Omega)$ the set of distributions f on $\partial\Omega$ such that Yf belongs to $\mathscr{H}^p(\partial\Omega)$ for any differential operator Y on $\partial\Omega$ of order less or equal than k. Then, we have the following theorem.

THEOREM 3.5. *The Szegö operator maps* $\mathscr{H}^p_k(\partial\Omega)$ *to itself and also to* $\mathscr{H}^p_k(\Omega)$ *boundedly for* $p_0 < p \leq 1$ *and* $k \in \mathbb{N}$. *In particular, this proves that* $\mathscr{H}^p_k(\partial\Omega) \cap \mathscr{H}(\Omega)$ *is included in* $\mathscr{H}^p_k(\Omega)$.

Remark. The converse inclusion should be also true. In [D1], this result is proved for strictly pseudoconvex domains and domains of finite type in $\mathbb{C}^2$. A similar proof must work in decoupled domains since, in this case, the geometry is well understood [D2].

Proof. First, let us assume that $k = 0$. The boundedness from $\mathscr{H}^p(\partial\Omega)$ to itself follows from Theorem 2.2. Now, let $f \in \mathscr{H}^p(\partial\Omega)$ with $p_0 < p \leq 1$, we want to show that

$$\sup_{0<\varepsilon<1} \int_{\partial\Omega_\varepsilon} |S(f)(\zeta)|^p \, d\sigma_\varepsilon(\zeta) \leq C\|f\|_{\mathscr{H}^p(\partial\Omega)}.$$

Since f has an atomic decomposition, it suffices to prove that for any atom a, we have

$$\sup_{0<\varepsilon<1} \int_{\partial\Omega_\varepsilon} |S(a)(\zeta)|^p \, d\sigma_\varepsilon(\zeta) \leq C.$$

Let $z \in \partial\Omega_\varepsilon$, we write $z = \mathbf{n}(x, \varepsilon)$ with $x \in \partial\Omega$. So

$$S(a)(z) = S(a)(\mathbf{n}(x, \varepsilon)) = S_\varepsilon(a)(x) = \int_{\partial\Omega} S(\mathbf{n}(x, \varepsilon), y) a(y) \, d\sigma(y).$$

We want to show that $x \to S_\varepsilon(a)(x) \in L^p(\partial\Omega)$ for every $0 < \varepsilon < 1$, independently of ε.

By the same method as the one used in the proof of Proposition 3.1 and with the same notations, we have

$$S_\varepsilon = S + \sum_{j=1}^{n} (C_j - C_j^\varepsilon)(r_j S),$$

where C_j^ε satisfy uniform estimates. So, S_ε is an operator of order 0 with uniform estimates.

Suppose that a is supported in $\tilde{B}(x_0, \delta)$ with $\|a\|_{L^2(\partial\Omega)} \leq c\sigma(\tilde{B}(x_0, \delta))^{1/2 - 1/p}$. We write

$$\int_{\partial\Omega} |S_\varepsilon(a)(x)|^p \, d\sigma(x) = \int_{\rho(x,x_0)\leq\delta} + \int_{\rho(x,x_0)\geq\delta} = \mathrm{I} + \mathrm{II},$$

$$\begin{aligned}
\mathrm{I} &= \int_{\rho(x,x_0)\leq\delta} \left| \int_{\tilde{B}(x_0,\delta)} S(\mathbf{n}(x, \varepsilon), y) a(y) \, d\sigma(y) \right|^p d\sigma(x) \\
&\leq \left(\int_{\tilde{B}(x_0,\delta)} |S_\varepsilon(a)(x)|^2 \, d\sigma(x) \right)^{p/2} \times \left(\int_{\tilde{B}(x_0,\delta)} d\sigma(x) \right)^{1-p/2} \\
&\leq C\|a\|^p_{L^2(\partial\Omega)} \times \sigma(\tilde{B}(x_0, \delta))^{1-p/2} \\
&\leq C
\end{aligned}$$

since S_ε is an operator of order 0 and S_ε maps $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$ independently of ε.

For the second part, we write

$$\{x \in \partial\Omega, \rho(x, x_0) \geq \delta\} = \bigcup_{k>1} I_k,$$

where

$$\mathrm{I}_k = \{x \in \partial\Omega, 2^k\delta \leq \rho(x, x_0) \leq 2^{k+1}\delta\}.$$

Then,

$$\int_{\rho(x,x_0)\geq 0} |S_\varepsilon(a)(x)| \, d\sigma(x) = \sum_k \int_{I_k} |L(x)| \, d\sigma(x),$$

where

$$L(x) = \int_{\tilde{B}(x_0\delta)} (S(\mathbf{n}(x, \varepsilon), y) - S(\mathbf{n}(x, \varepsilon), x_0))a(y)\, d\sigma(y)$$

(by the moment condition satisfied by a).

For any $k > 1$,

$$\begin{aligned}\int_{I_k} |L(x)|\, d\sigma(x) &\le \int_{I_k}\int_{\tilde{B}(x_0,\delta)} |S(\mathbf{n}(x, \varepsilon), y) \\ &\qquad - S(\mathbf{n}(x, \varepsilon), x_0)| \cdot |a(y)|\, d\sigma(y)\, d\sigma(x) \\ &\le \int_{\tilde{B}(x_0,\delta)} |a(y)| \int_{I_k} \left(\frac{\rho(x_0, y)}{\rho(x_0, x)}\right. \\ &\qquad \left. + \sum_j \frac{\mu_j(x_0, \rho(x_0, y))}{\mu_j(x_0, \rho(x_0, x))}\right) \frac{d\sigma(x)}{V(x_0, x)}\, d\sigma(y) \\ &\le \int_{\tilde{B}(x_0,\delta)} |a(y)| \int_{I_k \cap \{\rho(x, x_0) \ge 2^k \rho(y, x_0)\}} \\ &\qquad \left(\frac{1}{2^k} + \sum_j \frac{1}{2^{k/\alpha_j}}\right) \frac{d\sigma(x)}{V(x, x_0)}\, d\sigma(y) \\ &\le \left(\sum_j \frac{1}{2^{k/\alpha_j}}\right) \cdot \int_{\tilde{B}(x_0,\delta)} |a(y)|\, d\sigma(y) \\ &\le \left(\sum_j \frac{1}{2^{k/\alpha_j}}\right) \cdot \|a\|_{L^2(\partial\Omega)} \times \sigma(\tilde{B}(x_0, \delta))^{1/2} \\ &\le \left(\sum_j \frac{1}{2^{k/\alpha_j}}\right) \times \sigma(\tilde{B}(x_0, \delta))^{1-1/p}.\end{aligned}$$

Thus

$$\begin{aligned}\left(\int_{\rho(x, x_0) \ge \delta} |S_\varepsilon(a)(x)|^p\, d\sigma(x)\right)^{1/p} &\le \left(\sum_k \int_{I_k} |L(x)|^p\, d\sigma(x)\right)^{1/p} \\ &\le \sum_k \sigma(I_k)^{1/p-1} \\ &\qquad \times \left(\sum_j \frac{1}{2^{k/\alpha_j}}\right) \times \sigma(\tilde{B}(x_0, \delta))^{1-1/p} \\ &\le C \sum 2^{nk(1/p-1)} \times \left(\sum_j \frac{1}{2^{k/\alpha_j}}\right).\end{aligned}$$

So, it converges if $1/p < 1 + 1/n\alpha_k$ for $k = 1$ to $n - 1$. So, it suffices to choose p_0 such that $1/p_0 \leq 1 + 1/n\alpha_k$ for $k = 1$ to $n - 1$.

Now, we have to consider the case $k \neq 0$. Clearly the commutation properties satisfied by S allow to show that S is bounded from $\mathcal{H}_k^p(\partial\Omega)$ to itself. So, it is sufficient to show that a holomorphic function in $\mathcal{H}_k^p(\partial\Omega)$ is also in $\mathcal{H}_k^p(\Omega)$. To simplify, let us assume that $k = 1$.

So, let F be holomorphic in Ω and denote by f the value of F on the boundary in the distribution sense. By assumption, f, Nf, and Z_jf for $j = 1$ to $n - 1$ are in $\mathcal{H}^p(\partial\Omega)$. Since F is holomorphic in Ω, we can conclude that each derivative $\partial f/\partial z_j \in \mathcal{H}^p(\partial\Omega)$ (since $\partial F/\partial z_j$ is a linear combination of Z_jF, $j = 1$ to $n - 1$, and $NF = \partial F/\partial z_n$. So, it suffices to verify that

$$\frac{\partial F}{\partial z_j} = S\left(\frac{\partial F}{\partial z_j}\right);$$

then we will be able to conclude since we have shown that S maps $\mathcal{H}^p(\partial\Omega)$ into $\mathcal{H}^p(\Omega)$. These two functions are holomorphic by definition and have the same values on $\partial\Omega$ in the distribution sense. In particular, they are harmonic, and we can conclude that they are identical. ∎

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